Motives for elliptic modular groups

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Abstract

In order to study the arithmetic structure of elliptic modular groups which are the fundamental groups of compactified modular curves, these truncated group algebras and their direct sums are considered as elliptic modular motives. Our main result is a new theory of Hecke operators on these motives which gives a congruence relation to the Galois action, and a motivic decomposition to Hecke components on which Hecke operators act as scalar plus nilpotent matrices. As its application, we show that these motives become the direct sums of pure motives over certain number fields, and describe iterated Shimura integrals by the products of ordinary period integrals. Further, we give a description of these motives as the spaces of noncommutative modular symbols with Hecke action.

1. Introduction

Under the influence of Grothendieck's consideration on motives, Deligne [D2] started the motivic theory of the fundamental groups $\pi_1(X)$ of algebraic varieties X, and he showed that the motives for $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$ have rich structure as mixed Tate motives. The aim of this paper is to study *elliptic modular motives* constructed from the elliptic modular group $\pi_1(\overline{M}_n(\mathbb{C}))$, where $\overline{M}_n$ is the compactified modular curve over $\mathbb{Z}[1/n]$ of level $n \geq 3$.

In order to construct motives over $\mathbb{Q}$ according to Deligne's theory, we define the Betti realization of an elliptic modular motive as the direct sum of the truncated group algebras of $\pi_1(\overline{M}_n(\mathbb{C}))$ whose base points run through the cusps on $\overline{M}_n$. This l-adic structure with action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$ is derived from the natural $\mathbb{Z}[1/n]$-scheme structure on $\overline{M}_n$, and the Hodge structure is described by iterated integrals of 1-forms on $\overline{M}_n(\mathbb{C})$. This motive has mixed structure whose pure components are subquotients of $H_1(\overline{M}_n(\mathbb{C}); \mathbb{Q})^{\otimes i}$, and hence these components consist of tensor products of the dual motives attached to cusp forms of weight 2 of level n.

Our main result is to construct a new theory of Hecke operators which act on this elliptic modular motive and satisfy a congruence relation to the Galois action. The geometric congruence relation, which was given by Eichler, Shimura

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[S] and completed by Deligne [D1], is already essential to define the Hecke action before showing the congruence relation. This Hecke theory has the following applications:

- Our Hecke action preserves the motivic structure of the elliptic modular motive, and iterated integrals of holomorphic 1-forms on $\overline{M}_n(\mathbb{C})$, called iterated Shimura integrals of weight 2 (cf. [M1, 2]), are regarded as periods of this motive. Therefore, one can see that there is Hecke action on the space of these iterated integrals which gives a partial solution to a problem posed in [M2, 3.3] defining such action on the space of iterated Shimura integrals of any weight.

- The elliptic modular motive is decomposed to the direct sum of submotives called *Hecke components* corresponding to scalar-valued representations of the Hecke algebra. On these Hecke components, Hecke operators act as scalar plus nilpotent matrices, and hence the study of the motivic structure becomes easier. In particular, using the above noncommutative congruence relation, we will show that this motive becomes the direct sum of pure motives over a number field, and describe iterated Shimura integrals of weight 2 by the products of period integrals.

The last assertion is applied to expressing multiple L-values of cusp forms of weight 2 by their ordinary L-values.

Furthermore, we describe the Betti realization of the elliptic modular motive as the space of *noncommutative modular symbols*, and show that the Hecke action can be computed using these symbols.

2. Elliptic modular motives

2.1. In this section, we construct motives for elliptic modular groups based on Deligne's theory [D1]. Let π_1 be the topological fundamental group $\pi_1(X; x)$ of a Riemann surface X with base point x. Then π_1 is finitely generated. Denote by $\mathbb{Q}[\pi_1]$ the group $\mathbb{Q}$-algebra of π_1 with the augmentation ideal I, and by

$$\mathbb{Q}[\pi_1]^\wedge = \varprojlim \mathbb{Q}[\pi_1]/I^m,$$

the completion of $\mathbb{Q}[\pi_1]$ with the ideal $\widehat{I}$ given as the completion of I. Let

$$\Delta : \mathbb{Q}[\pi_1]^\wedge \to \mathbb{Q}[\pi_1]^\wedge \,\widehat{\otimes}\, \mathbb{Q}[\pi_1]^\wedge$$

be the diagonal homomorphism which is a continuous algebra homomorphism induced from $g \mapsto g \otimes g$ $(g \in \pi_1)$. Then the Malcev Lie algebra $\mathrm{Lie}(\pi_1)$ of π_1 is defined as the set

$$\left\{ a \in \mathbb{Q}\,[\pi_1]^\wedge \mid \Delta(a) = 1 \otimes a + a \otimes 1 \right\}$$

of primitive elements in which the bracket product $[a, b]$ is given by $ab - ba$. Then the exponential map

$$\exp(a) = \sum_{i=0}^{\infty} \frac{a^i}{i!}$$

gives a bijection from $\mathrm{Lie}(\pi_1)$ onto the set

$$\left\{ g \in \mathbb{Q}\,[\pi_1]^\wedge \,\middle|\, g - 1 \in \widehat{I},\ \Delta(g) = g \otimes g \right\}$$

of grouplike elements, and its inverse map is the logarithmic map

$$\log(g) = -\sum_{i=1}^{\infty} \frac{(1-g)^i}{i}.$$

For a positive integer N, let

$$A_N(X; x) = \mathbb{Q}\,[\pi_1(X; x)]\,/I^{N+1}$$

be the Nth truncated group algebra of $\pi_1(X; x)$ over $\mathbb{Q}$. Then the Nth truncated Malcev Lie algebra $\mathrm{Lie}_N(\pi_1) = \mathrm{Lie}_N(\pi_1(X; x))$ is defined as the image of $\mathrm{Lie}(\pi_1(X; x))$ into $A_N(X; x)$ by the natural projection. The corresponding unipotent algebraic group $G_N = G_N(\pi_1(X; x))$ over $\mathbb{Q}$ is characterized by that $G_N(\mathbb{Q})$ consists of the images of grouplike elements into $A_N(X; x)$ by the natural projection. Further, for any field K of characteristic 0, the exponential and logarithmic maps give bijections $\mathrm{Lie}_N(\pi_1) \otimes K \xleftrightarrow{\ \sim\ } G_N(K)$ which are inverse to each other.

PROPOSITION 2.1. *If $g_1, g_2 \in G_N(K)$ satisfy that $g_1^t = g_2^t$ for some positive integer t, then $g_1 = g_2$.*

Proof. By the assumption,

$$g_1 = \exp\left(\frac{\log(g_1^t)}{t}\right) = \exp\left(\frac{\log(g_2^t)}{t}\right) = g_2$$

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which completes the proof. $\square$

For a prime l, denote by π_1^l the l-adic (algebraic) fundamental group. For results on this subject, see [G]. Then for $x \in X$, the l-adic completion of $\pi_1(X; x)$ is canonically isomorphic to $\pi_1^l(X; x)$, and hence the natural group homomorphism

$$\phi_X : \pi_1(X; x) \to G_N(\mathbb{Q}) \subset A_N(X; x)$$

gives rise to a group homomorphism

$$\pi_1^l(X; x) \to G_N(\mathbb{Q}_l) \subset A_N(X; x) \otimes \mathbb{Q}_l$$

which we denote by the same symbol.

2.2. Let n be an integer ≥ 3, denote by P_n the set of primitive n-th roots of 1, and by R_n the subring $\mathbb{Z}[1/n, \zeta]$ of $\mathbb{C}$, where ζ is an element of P_n. Let M_n be the affine modular curve over $\mathbb{Z}[1/n]$ which classifies elliptic curves E with level n structure, i.e., isomorphism

$$\lambda : (\mathbb{Z}/n\mathbb{Z})^{\oplus 2} \xrightarrow{\sim} E[n] \overset{\text{def}}{=} \operatorname{Ker}(n : E \to E).$$

For each $\zeta \in P_n$, let M_ζ be the modular curve over R_n which classifies (E, λ) such that $e(a, b) = \zeta^{\psi(\lambda^{-1}(a), \lambda^{-1}(b))}$, where e denotes the Weil pairing, and ψ denotes the standard symplectic form. Then each M_ζ is geometrically irreducible, and

$$M_n \otimes R_n = \bigsqcup_{\zeta \in P_n} M_\zeta.$$

Further, $M_\zeta(\mathbb{C}) \cong H/\Gamma(n)$, where $H = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$ denotes the Poincaré upper half plane, and $\Gamma(n) = \operatorname{Ker}(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/n\mathbb{Z}))$ denotes the principal congruence subgroup of level n. Let $\overline{M}_\zeta = M_\zeta \cup C_\zeta$ be the compactified modular curve over R_n, where C_ζ denotes the scheme of cusps which is finite and étale over R_n and consisting of R_n-rational points (see [I]). Then $\overline{M}_\zeta(\mathbb{C}) \cong \overline{H}/\Gamma(n)$, where $\overline{H} = H \cup \mathbb{P}^1(\mathbb{Q})$, and there is a unique proper smooth curve $\overline{M}_n$ over $\mathbb{Z}[1/n]$ such that $\overline{M}_n \otimes R_n = \bigsqcup_{\zeta \in P_n} \overline{M}_\zeta$.

In what follows, fix a positive integer N. Then the Betti realization $V = V_N$ of the Nth *truncated elliptic modular motive of level* n is defined as the direct sum of the Nth truncated group algebra of $\pi_1(\overline{M}_\zeta(\mathbb{C}); \alpha)$, where ζ and α run through P_n and C_ζ respectively:

$$V = V_N = \bigoplus_{\zeta, \alpha} A_N\left(\overline{M}_\zeta(\mathbb{C}); \alpha\right).$$

We regard V together with the following structure as a motive.

First, we consider the l-adic structure on V, i.e., the $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$-module structure on $V \otimes \mathbb{Q}_l$. For $x \in \overline{M}_\zeta\left(\overline{\mathbb{Q}}\right)$, $\pi_1^l\left(\overline{M}_\zeta(\mathbb{C}); x\right) \cong \pi_1^l\left(\overline{M}_\zeta \otimes \overline{\mathbb{Q}}; x\right)$ is the automorphism group of the fiber systems $\left\{f_Y^{-1}(x)\right\}$ for Galois coverings $f_Y : Y \to \overline{M}_\zeta \otimes \overline{\mathbb{Q}}$ of l-power degree. Therefore, for $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$, the correspondence

$$\gamma \mapsto \sigma \circ \gamma \circ \sigma^{-1} \ \left(\gamma \in \pi_1^l\left(\overline{M}_\zeta(\mathbb{C}); x\right)\right)$$

together with $\phi_{\overline{M}_\zeta(\mathbb{C})}$ give a $\mathbb{Q}_l$-algebra homomorphism

$$\eta(\sigma) : A_N\left(\overline{M}_\zeta(\mathbb{C}); x\right) \otimes \mathbb{Q}_l \to A_N\left(\overline{M}_{\sigma(\zeta)}(\mathbb{C}); \sigma(x)\right) \otimes \mathbb{Q}_l.$$

Since $\bigsqcup_\zeta C_\zeta$ is stable under the Galois action over $\mathbb{Q}$, $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ acts naturally on $V \otimes \mathbb{Q}_l$.

Second, following Hain [H1, 2] (see also [Mo]), we consider weight and Hodge filtrations of V. Chen [C1, 2] showed that $A_N\left(\overline{M}_\zeta(\mathbb{C}); \alpha\right) \otimes \mathbb{C}$ is dual to the $\mathbb{C}$-vector space $B_N\left(\overline{M}_\zeta(\mathbb{C}); \alpha\right)$ consisting of finite sums of iterated integrals

$$\int w_1 \cdots w_r \ \left(w_i : \text{smooth 1-forms on } \overline{M}_\zeta(\mathbb{C}), \ r \leq N\right)$$

which are homotopy functional on {loops based at α}. Hence $V \otimes \mathbb{C}$ is dual to

$$B_N = \bigoplus_{\zeta, \alpha} B_N\left(\overline{M}_\zeta(\mathbb{C}); \alpha\right),$$

where ζ, α run through P_n, C_ζ respectively. Hain [H1, 2] defined the weight and Hodge filtrations of B_N as

$$W^l\left(B_N\right) \ = \ \begin{cases} B_l & (l \leq N), \\ B_N & (l \geq N), \end{cases}$$

$$F^k\left(B_N\right) \ = \ \bigoplus_{\zeta, \alpha} \left\{ \begin{array}{l} \text{sums of iterated integrals of 1-forms with } \geq k \ dz\text{'s,} \\ \text{where } z \text{ is a holomorphic coordinate on } \overline{M}_\zeta \end{array} \right\},$$

and hence by the duality, V has weight filtration and $V \otimes \mathbb{C}$ has Hodge filtration. Furthermore, Hain defined de Rham structure on V which is induced from the de Rham cohomology group $H_{\mathrm{DR}}^1\left(\overline{M}_\zeta; \mathbb{Q}(\zeta)\right)$ and is compatible with the Hodge filtration on $V \otimes \mathbb{C}$.

Finally, we only notice that the crystalline structure on V is induced from that on the first cohomology groups of $\overline{M}_\zeta$ ($\zeta \in P_n$) and is constructed by Shiho [S].

3. Congruence relation

3.1. For a subfield k of $\mathbb{C}$, an *algebraic correspondence* on $M_n \otimes k$ means a k-morphism $\tau_1 \times \tau_2$ from a 1-dimensional k-scheme Z to $(M_n \times M_n) \otimes k$ such that τ_1 and τ_2 are finite and flat surjective morphisms $Z \to M_n \otimes k$. Then for a prime l and a $\overline{k}$-rational point $z \in \overline{Z}(\overline{k})$ on the completion $\overline{Z}$ of Z, an algebraic correspondence $\tau_1 \times \tau_2 : Z \to (M_n \times M_n) \otimes k$ is called *compatible with π_1^l and z* if the following conditions are satisfied:

(3.1) $Z \otimes \overline{k}$ is a direct sum $\bigsqcup_{\zeta \in P_n} Z_{\zeta/\overline{k}}$ of connected and reduced schemes over $\overline{k}$ such that for $i = 1, 2$, $\tau_i(Z_{\zeta/\overline{k}}) = M_{\zeta_i} \otimes \overline{k}$ for some $\zeta_i \in P_n$.

(3.2) Denote by the same symbol the natural extension of τ_i from the completion $\overline{Z}_{\zeta/\overline{k}}$ of $Z_{\zeta/\overline{k}}$ into $\overline{M}_{\zeta_i} \otimes \overline{k}$, and denote by

$$\pi_1^l(\tau_i) : \pi_1^l\left(\overline{Z}_{\zeta/\overline{k}}; z\right) \to \pi_1^l\left(\overline{M}_{\zeta_i} \otimes \overline{k}; \tau_i(z)\right) \ \ (i = 1, 2)$$

the natural group homomorphisms. Then there exists a group homomorphism

$$\iota_z : \pi_1^l\left(\overline{M}_{\zeta_1} \otimes \overline{k}; \tau_1(z)\right) \to \pi_1^l\left(\overline{M}_{\zeta_2} \otimes \overline{k}; \tau_2(z)\right)$$

such that $\iota_z \circ \pi_1^l(\tau_1) = \pi_1^l(\tau_2)$.

Let $\tau_1 \times \tau_2 : Z \to (M_n \times M_n) \otimes k$ be an algebraic correspondence compatible with π_1^l and any point on $\overline{Z}_{\zeta/\overline{k}}(\overline{k})$. Since $\tau_1 : \overline{Z}_{\zeta/\overline{k}} \to \overline{M}_{\zeta_1} \otimes \overline{k}$ is finite, flat and surjective, the scheme theoretic fiber $\tau_1^{-1}(x)$ of $x \in \overline{M}_{\zeta_1}(\overline{k})$ is expressed as $\sum_i m_i \cdot z_i$ with $m_i \in \mathbb{N}$ and $z_i \in \overline{Z}_{\zeta/\overline{k}}(\overline{k})$ such that $\sum_i m_i = \deg(\tau_1)$. Then for $\gamma \in \pi_1^l\left(\overline{M}_{\zeta_1} \otimes \overline{k}; x\right)$, one can define

$$\begin{aligned}
E(Z)(\gamma) &= \sum_i m_i \cdot \left(\phi_{\overline{M}_{\zeta_2}(\mathbb{C})} \circ \iota_{z_i}\right)(\gamma) \\
&\in \bigoplus_i A_N\left(\overline{M}_{\zeta_2}(\mathbb{C}); \tau_2(z_i)\right) \otimes \mathbb{Q}_l.
\end{aligned}$$

Since each ι_{z_i} is a group homomorphism, $E(Z)$ gives rise to $\mathbb{Q}_l$-linear endomorphisms of the direct sums of $A_N\left(\overline{M}_\zeta(\mathbb{C}); x\right) \otimes \mathbb{Q}_l$ for $\zeta \in P_n$ and $x \in \overline{M}_\zeta\left(\overline{k}\right)$. This endomorphism is also denoted by $E(Z) = E(\tau_1 \times \tau_2)$.

PROPOSITION 3.1.   Let $\tau_1 \times \tau_2$, $\tau_1' \times \tau_2'$ be algebraic correspondences $Z, Z' \to (M_n \times M_n) \otimes k$ compatible with π_1^l and any point on $\overline{Z}\left(\overline{k}\right)$, $\overline{Z'}\left(\overline{k}\right)$ respectively. Then the product $Z \cdot Z'$ of these correspondences is also compatible with π_1^l and any point on $\overline{(Z \cdot Z')}\left(\overline{k}\right)$, and $E(Z') \circ E(Z) = E\left(Z \cdot Z'\right)$.

Proof. Since the product $Z \cdot Z'$ is defined as the fiber product of $\tau_2 : Z \to M_n \otimes k$ and $\tau_1' : Z' \to M_n \otimes k$ over $M_n \otimes k$, there are natural projections $\pi : Z \cdot Z' \to Z$ and $\pi' : Z \cdot Z' \to Z'$. Let ι, ι' be the homomorphisms given in (3.2) for $\tau_1 \times \tau_2$, $\tau_1' \times \tau_2'$ respectively. Then $\tau_2 \circ \pi = \tau_1' \circ \pi'$ on $\overline{(Z \cdot Z')}$, and hence $\iota'_{\pi'(z)} \circ \iota_{\pi(z)}$ satisfies (3.2) for any $z \in \overline{(Z \cdot Z')}\left(\overline{k}\right)$ because

$$\left(\iota'_{\pi'(z)} \circ \iota_{\pi(z)}\right) \circ \pi_1^l(\tau_1 \circ \pi) = \iota'_{\pi'(z)} \circ \pi_1^l(\tau_2 \circ \pi) = \iota'_{\pi'(z)} \circ \pi_1^l(\tau_1' \circ \pi') = \pi_1^l(\tau_2' \circ \pi').$$

For $x \in \overline{M}_{\zeta_1}\left(\overline{k}\right)$, let $\tau_1^{-1}(x) = \sum_i m_i \cdot z_i$ and $\pi^{-1}(z_i) = \sum_j n_{ij} \cdot w_{ij}$, where $\sum_i m_i = \deg(\tau_1)$ and $\sum_j n_{ij} = \deg(\pi) = \deg(\tau_1')$. Then $(\tau_1')^{-1}(\tau_2(z_i)) = \sum_j n_{ij} \cdot \pi'(w_{ij})$, and hence for $\gamma \in \pi_1^l\left(\overline{M}_{\zeta_1} \otimes \overline{k}; x\right)$,

$$
\begin{aligned}
\left(E(Z') \circ E(Z)\right)(\gamma) &= E(Z')\left(\sum_i m_i \cdot \left(\phi \circ \iota_{z_i}\right)(\gamma)\right) \\
&= \sum_{i,j} (n_{ij} m_i) \cdot \left(\phi \circ \iota'_{\pi'(w_{ij})} \circ \iota_{z_i}\right)(\gamma) \\
&= E(Z \cdot Z')(\gamma)
\end{aligned}
$$

which implies that $E(Z') \circ E(Z) = E\left(Z \cdot Z'\right)$ holds on the direct sums of $A_N\left(\overline{M}_\zeta(\mathbb{C}); x\right) \otimes \mathbb{Q}_l$ for $\zeta \in P_n$ and $x \in \overline{M}_\zeta\left(\overline{k}\right)$. $\square$

3.2.   Let $g \in GL_2(\mathbb{Q})$ has integral entries and positive determinant d prime to n. Then $\Gamma(n, g) = \Gamma(n) \cap g^{-1}\Gamma(n)g$ is finite index in $\Gamma(n)$ and in $g^{-1}\Gamma(n)g$. Under an identification $M_\zeta(\mathbb{C}) = H/\Gamma(n)$, there exists a natural model Z_ζ of $H/\Gamma(n, g)$ over $\overline{\mathbb{Q}}$ because $\Gamma(n, g) \supset \Gamma(\det(g) \cdot n)$. Further, let

$$\tau_1 : Z_\zeta(\mathbb{C}) = H/\Gamma(n, g) \to M_\zeta(\mathbb{C}) = H/\Gamma(n)$$

be the natural surjection which we denote by $\pi(g)$, and let $\tau_2 : Z_\zeta(\mathbb{C}) \to M_\zeta(\mathbb{C})$ be the composite

$$Z_\zeta(\mathbb{C}) = H/\Gamma(n, g) \xrightarrow{g} H /\Gamma\left(n, g^{-1}\right) \xrightarrow{\pi(g^{-1})} M_{\zeta^d}(\mathbb{C}).$$

Then $\tau_1 \times \tau_2 : Z(\mathbb{C}) = \bigsqcup_{\zeta \in P_n} Z_\zeta(\mathbb{C}) \to M_n(\mathbb{C}) \times M_n(\mathbb{C})$ gives an algebraic correspondence on $M_n \otimes \mathbb{C}$ called a *Hecke correspondence* and denoted by $T(g)$. Since $\overline{M}_\zeta(\mathbb{C}) = \overline{H}/\Gamma(n)$ and $\overline{Z}_\zeta(\mathbb{C}) = \overline{H}/\Gamma(n, g)$,

$$\tau_1^{-1}\left(C_\zeta\right) = \tau_2^{-1}\left(C_{\zeta^d}\right) = \overline{Z}_\zeta - Z_\zeta.$$

PROPOSITION 3.2. *For* $i = 1, 2$ *and* $z \in \overline{Z}_\zeta(\mathbb{C})$, *the* $\mathbb{Q}$-*algebra homomorphism*

$$A_N(\tau_i) : A_N\left(\overline{Z}_\zeta(\mathbb{C}); z\right) \to A_N\left(\overline{M}_{\zeta_i}(\mathbb{C}); \tau_i(z)\right)\ \left(\zeta_1 = \zeta, \zeta_2 = \zeta^d\right)$$

induced from the natural group homomorphism

$$\pi_1(\tau_i) : \pi_1\left(\overline{Z}_\zeta(\mathbb{C}); z\right) \to \pi_1\left(\overline{M}_{\zeta_i}(\mathbb{C}); \tau_i(z)\right)$$

is surjective.

Proof. We may prove the assertion for $i = 1$. Take $\gamma \in \pi_1\left(\overline{M}_\zeta(\mathbb{C}); \tau_1(z)\right)$. Since $\pi_1\left(Z_\zeta(\mathbb{C})\right) = \Gamma(n, g)$ has finite index in $\pi_1\left(M_\zeta(\mathbb{C})\right) = \Gamma(n)$ and the natural homomorphism $\pi_1\left(M_\zeta(\mathbb{C}); \tau_1(z)\right) \to \pi_1\left(\overline{M}_\zeta(\mathbb{C}); \tau_1(z)\right)$ (if $\tau_1(z) \notin M_\zeta$, then we consider $\tau_1(z)$ as a tangential base point) is surjective, there are a positive integer t and $\widetilde{\gamma} \in \pi_1\left(\overline{Z}_\zeta(\mathbb{C}); z\right)$ such that $\gamma^t = \pi_1(\tau_1)\left(\widetilde{\gamma}\right)$. Since $A_N(\tau_1)$ is a $\mathbb{Q}$-algebra homomorphism,

$$\begin{aligned}
\gamma &= \exp\left(\frac{1}{t}\log\left(\left(\phi_{\overline{M}_{\zeta_1}(\mathbb{C})} \circ \pi_1(\tau_1)\right)(\widetilde{\gamma})\right)\right) \\
&= A_N(\tau_1)\left(\exp\left(\frac{1}{t}\log\left(\phi_{\overline{Z}_\zeta(\mathbb{C})}(\widetilde{\gamma})\right)\right)\right)
\end{aligned}$$

belongs to the image of $A(\tau_1)$, and hence this map is surjective. $\square$

PROPOSITION 3.3. *Assume that* $T(g)$ *is compatible with* π_1^l *and a point* z_0 *on* $\overline{Z}_\zeta(\mathbb{C})$. *Then we have:*

(1) *Let z be any point on $\overline{Z}_\zeta(\mathbb{C})$. Then $T(g)$ is compatible with π_1^l and z, and $\phi \circ \iota_z$ gives a unique $\mathbb{Q}$-algebra homomorphism*

$$A_N(\iota_z) : A_N\left(\overline{M}_\zeta(\mathbb{C}); \tau_1(z)\right) \to A_N\left(\overline{M}_{\zeta^d}(\mathbb{C}); \tau_2(z)\right)$$

satisfying that $A_N(\iota_z) \circ A_N(\tau_1) = A_N(\tau_2)$.

(2) $E\left(T(g)\right)$ *gives naturally a $\mathbb{Q}$-linear endomorphism of*

$$V = \bigoplus_{\zeta,\alpha} A_N\left(\overline{M}_\zeta(\mathbb{C}); \alpha\right).$$

(3) *If $T(g)$ is an algebraic correspondence on $M_n \otimes \mathbb{Q}$, then $\eta(\sigma) \circ E\left(T(g)\right) = E\left(T(g)\right) \circ \eta(\sigma)$ on $V \otimes \mathbb{Q}_l$ for any $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$.*

Proof. First, we prove (1). For the proper smooth curves $X = \overline{Z}_\zeta \otimes \mathbb{C}$ and $\overline{M}_\zeta \otimes \mathbb{C}$ over $\mathbb{C}$, and a point $x \in X(\mathbb{C})$, note that $\pi_1^l(X; x) \cong \pi_1^l(X(\mathbb{C}); x)$ is the automorphism group of the fiber system of x on pro-l Galois coverings of $X(\mathbb{C})$. Therefore, taking inner automorphisms given by a path connecting z_0, z and their images by τ_i $(i = 1, 2)$, one can see that $T(g)$ is compatible with π_1^l and z. The uniqueness of such a $\mathbb{Q}$-algebra homomorphism follows from Proposition 3.2, and we prove the existence. For $\gamma \in \pi_1\left(\overline{M}_\zeta(\mathbb{C}); \tau_1(z)\right)$, as seen in the proof of Proposition 3.2, there are a positive integer t and $\widetilde{\gamma} \in \pi_1\left(\overline{Z}_\zeta(\mathbb{C}); z\right)$ such that $\gamma^t = \pi_1(\tau_1)(\widetilde{\gamma})$. Then by (3.2),

$$
\begin{aligned}
\left(\phi_{\overline{M}_{\zeta^d}(\mathbb{C})} \circ \iota_z\right)(\gamma)^t &= \left(\phi_{\overline{M}_{\zeta^d}(\mathbb{C})} \circ \pi_1(\tau_2)\right)(\widetilde{\gamma}) \\
&= \exp\left(\frac{1}{t}\log\left(\left(\phi_{\overline{M}_{\zeta^d}(\mathbb{C})} \circ \pi_1(\tau_2)\right)(\widetilde{\gamma})\right)\right)^t,
\end{aligned}
$$

and hence by Proposition 2.1, we have

$$
\begin{aligned}
\left(\phi_{\overline{M}_{\zeta^d}(\mathbb{C})} \circ \iota_z\right)(\gamma) &= \exp\left(\frac{1}{t}\log\left(\left(\phi_{\overline{M}_{\zeta^d}(\mathbb{C})} \circ \pi_1(\tau_2)\right)(\widetilde{\gamma})\right)\right) \\
&\in A_N\left(\overline{M}_{\zeta^d}(\mathbb{C}); \tau_2(z)\right).
\end{aligned}
$$

Since $\phi_{\overline{M}_{\zeta^d}(\mathbb{C})} \circ \iota_z$ is a group homomorphism, it gives a $\mathbb{Q}$-algebra homomorphism

$$A_N(\iota_z) : A_N\left(\overline{M}_\zeta(\mathbb{C}); \tau_1(z)\right) \to A_N\left(\overline{M}_{\zeta^d}(\mathbb{C}); \tau_2(z)\right)$$

which satisfies that

$$(A_N(\iota_z) \circ A_N(\tau_1)) \left(\exp \left(\frac{1}{t} \log \left(\phi_{\overline{Z}_\zeta(\mathbb{C})}(\widetilde{\gamma}) \right) \right) \right)$$
$$= A_N(\iota_z)(\gamma)$$
$$= A_N(\tau_2) \left(\exp \left(\frac{1}{t} \log \left(\phi_{\overline{Z}_\zeta(\mathbb{C})}(\widetilde{\gamma}) \right) \right) \right),$$

and that $A_N(\iota_z) \circ A_N(\tau_1) = A_N(\tau_2)$ generally.

Second, the assertion (2) follows from (1) because $\tau_2 \left(\tau_1^{-1}(C_\zeta) \right) \subset C_{\zeta^d}$, and then we prove (3). Take $\alpha \in C_\zeta$ with fiber $\tau_1^{-1}(\alpha) = \sum_i m_i \cdot \beta_i$ $\left(\beta_i \in \overline{Z}_\zeta - Z_\zeta \right)$ and $\gamma \in \pi_1 \left(\overline{M}_\zeta(\mathbb{C}); \alpha \right)$. Since τ_1 and τ_2 are $\mathbb{Q}$-morphisms, by the expression

$$\left(\phi_{\overline{M}_{\zeta^d}(\mathbb{C})} \circ \iota_{\beta_i} \right)(\gamma) = \exp \left(\frac{1}{t} \log \left(\left(\phi_{\overline{M}_{\zeta^d}(\mathbb{C})} \circ \pi_1(\tau_2) \right)(\widetilde{\gamma}) \right) \right),$$

we have

$$(\eta(\sigma) \circ E(T(g)))(\gamma) = \eta(\sigma) \left(\sum_i m_i \cdot \left(\phi_{\overline{M}_{\zeta^d}(\mathbb{C})} \circ \iota_{\beta_i} \right)(\gamma) \right)$$
$$= \sum_i m_i \cdot \left(\phi_{\overline{M}_{\sigma(\zeta)^d}(\mathbb{C})} \circ \iota_{\sigma(\beta_i)} \circ \eta(\sigma) \right)(\gamma)$$
$$= (E(T(g)) \circ \eta(\sigma))(\gamma).$$

Therefore, the commutativity of $\eta(\sigma)$ and $E(T(g))$ also holds on $V \otimes \mathbb{Q}_l$. $\square$

3.3. We will prove that some Hecke correspondences are compatible with π_1^l, and give endomorphisms of V satisfying the congruence relation. Let p be a prime not dividing n, and let $\overline{R}$ be a subring of $\overline{\mathbb{Q}}$ containing R_n such that $\overline{R}$ is a Henselian ring with residue field $\overline{\mathbb{F}}_p$. Since $\overline{M}_\zeta$ is proper smooth over R_n, by a result in [G], for any prime $l \neq p$,

$$\pi_1^l \left(\overline{M}_\zeta \otimes \overline{\mathbb{Q}} \right) \cong \pi_1^l \left(\overline{M}_\zeta \otimes \overline{R} \right) \cong \pi_1^l \left(\overline{M}_\zeta \otimes \overline{\mathbb{F}}_p \right),$$

and hence the p-th power (absolute Frobenius) morphism gives a group isomorphism $\pi_1^l \left(\overline{M}_\zeta \otimes \overline{\mathbb{Q}} \right) \to \pi_1^l \left(\overline{M}_{\zeta^p} \otimes \overline{\mathbb{Q}} \right)$ which we denote by F_p.

We recall results in [D1, §4] and [KM, §6]. Let $T(p,p) : M_n \to M_n \times M_n$ be the morphism over $\mathbb{Z}[1/n]$ sending (E, λ) to $((E, \lambda), (E, p\lambda))$. Then $T(p,p)$

gives an algebraic correspondence on $M_n \otimes \mathbb{Q}$, and $T(p,p) \otimes \mathbb{C} = T\left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\right)$.

Let $M_{n,p}$ be the moduli scheme over $\mathbb{Z}[1/n]$ classifying $((E_i, \lambda_i)_{i=1,2}, \varphi)$, where E_i are elliptic curves with level n structure λ_i, and $\varphi : E_1 \to E_2$ are isogenies of degree p such that $\lambda_2 = \varphi \circ \lambda_1$. Then $M_{n,p}$ becomes a regular scheme. For each $i = 1, 2$, let $\rho_i : M_{n,p} \to M_n$ be the $\mathbb{Z}[1/n]$-morphism sending $\varphi : (E_1, \lambda_1) \to (E_2, \lambda_2)$ to (E_i, λ_i). Then ρ_i is a finite and flat surjective morphism, and hence $\rho_1 \times \rho_2 : M_{n,p} \to M_n \times M_n$ gives an algebraic correspondence on $M_n \otimes \mathbb{Q}$ which we denote by $T(p)$. By definition, $T(p) \otimes \mathbb{C} = T\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right)$. Let Φ_1 and Φ_2 be the morphisms $M_n \otimes \mathbb{F}_p \to M_{n,p} \otimes \mathbb{F}_p$ sending (E, λ) to the Frobenius $F : (E, \lambda) \to \left(E^{(p)}, \lambda^{(p)}\right)$ and to the Vershiebung $V : \left(E^{(p)}, p\lambda^{(p)}\right) \to (E, \lambda)$ respectively. Then it is known as the geometric congruence relation that

$$\begin{cases} (\rho_1 \times \rho_2) \circ \Phi_1 & = \quad \text{id.} \times F, \\ (\rho_1 \times \rho_2) \circ \Phi_2 & = \quad (T(p,p)^{-1} \circ F) \times \text{id.}, \end{cases}$$

where $F : M_n \otimes \mathbb{F}_p \to M_n \otimes \mathbb{F}_p$ denotes the Frobenius morphism.

PROPOSITION 3.4. *For a prime $l \neq p$, the algebraic correspondences $T(p,p)$ and $T(p)$ on $M_n \otimes \mathbb{Q}$ are compatible with π_1^l and any $z \in M_n\left(\overline{R}\right)$.*

Proof. The compatibility of $T(p,p)$ with π_1^l is clear because the correspondence $(E, \lambda) \mapsto (E, p\lambda)$ gives an automorphism of M_n sending its cusps to those bijectively. For each $\zeta \in P_n$, let $M_{\zeta,p}$ be the moduli scheme over R_n classifying isogenies $\varphi : (E_1, \lambda_1) \to (E_2, \lambda_2)$ of degree p such that $(E_1, \lambda_1) \in M_\zeta$ and $(E_2, \lambda_2) \in M_{\zeta^p}$. Then $M_{n,p} \otimes R_n = \bigsqcup_{\zeta \in P_n} M_{\zeta,p}$ and $M_{\zeta,p} \otimes \overline{\mathbb{Q}}$ is irreducible. Hence (3.1) is satisfied for $Z_{\zeta/\overline{k}} = M_{\zeta,p} \otimes \overline{\mathbb{Q}}$ and $\tau_i = \rho_i$. Further, $M_{\zeta,p} \otimes \overline{\mathbb{F}}_p$ is a connected and reduced scheme obtained as the union of $\Phi_1\left(M_\zeta \otimes \overline{\mathbb{F}}_p\right)$ and $\Phi_2\left(M_{\zeta^p} \otimes \overline{\mathbb{F}}_p\right)$ at the supersingular points. Let $\overline{M}_{\zeta,p}$ be the normalization of $\overline{M}_\zeta$ in the function field of $M_{\zeta,p}$. Then $\overline{M}_{\zeta,p}$ is proper over R_n, and hence $\pi_1^l\left(\overline{M}_{\zeta,p} \otimes \overline{R}\right)$ is naturally isomorphic to $\pi_1^l\left(\overline{M}_{\zeta,p} \otimes \overline{\mathbb{F}}_p\right)$. Let $\overline{\rho}_1 : \overline{M}_{\zeta,p} \to \overline{M}_\zeta$ be the natural projection which is a unique extension of $\rho_1 : M_{\zeta,p} \to M_\zeta$. Further, $\overline{M}_{\zeta,p}$ is normal and $\overline{M}_{\zeta^p}$ is proper over R_n, hence there exists a unique morphism $\overline{\rho}_2 : \overline{M}_{\zeta,p} \to \overline{M}_{\zeta^p}$ which extends $\rho_2 : M_{\zeta,p} \to M_{\zeta^p}$ and is induced from the Neron model of the corresponding elliptic curve over $M_{\zeta,p}$ with level n structure. Note that $\Phi_1\left(M_\zeta\left(\overline{\mathbb{F}}_p\right)\right) \cup \Phi_2\left(M_{\zeta^p}\left(\overline{\mathbb{F}}_p\right)\right) = M_{\zeta,p}\left(\overline{\mathbb{F}}_p\right)$, and let ι_z be either F_p or $F_p^{-1} \circ E\left(T(p,p)\right)$ when the reduction of z belongs to either $\Phi_1\left(M_\zeta\left(\overline{\mathbb{F}}_p\right)\right)$ or

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$\Phi_2 \left(M_{\zeta^p} \left(\overline{\mathbb{F}}_p \right) \right) - \Phi_1 \left(M_\zeta \left(\overline{\mathbb{F}}_p \right) \right)$ respectively. Then ι_z gives group homomorphisms

$$\pi_1^l \left(M_\zeta \otimes \overline{\mathbb{F}}_p; \rho_1(z) \right) \to \pi_1^l \left(M_{\zeta^p} \otimes \overline{\mathbb{F}}_p; \rho_2(z) \right),$$

and

$$\pi_1^l \left(\overline{M}_\zeta \otimes \overline{\mathbb{F}}_p; \rho_1(z) \right) \cong \pi_1^l \left(\overline{M}_\zeta \otimes \overline{\mathbb{Q}}; \rho_1(z) \right)$$
$$\to \; \pi_1^l \left(\overline{M}_{\zeta^p} \otimes \overline{\mathbb{F}}_p; \rho_2(z) \right) \cong \pi_1^l \left(\overline{M}_{\zeta^p} \otimes \overline{\mathbb{Q}}; \rho_2(z) \right).$$

Further, by the geometric congruence relation, $\iota_z \circ \pi_1^l (\rho_1) = \pi_1^l (\rho_2)$ holds on $\pi_1^l \left(M_{\zeta,p} \otimes \overline{\mathbb{F}}_p; z \right)$. Therefore, $\iota_z \circ \pi_1^l (\overline{\rho}_1) = \pi_1^l (\overline{\rho}_2)$ holds on

$$\pi_1^l \left(\overline{M}_{\zeta,p} \otimes \overline{\mathbb{F}}_p; z \right) \cong \pi_1^l \left(\overline{M}_{\zeta,p} \otimes \overline{R}; z \right),$$

and hence on $\pi_1^l \left(\overline{M}_{\zeta,p} \otimes \overline{\mathbb{Q}}; z \right)$ via the natural group homomorphism

$$\pi_1^l \left(\overline{M}_{\zeta,p} \otimes \overline{\mathbb{Q}}; z \right) \to \pi_1^l \left(\overline{M}_{\zeta,p} \otimes \overline{R}; z \right).$$

Therefore, (3.2) is satisfied. $\square$

THEOREM 3.5. *Let p be a prime not dividing n.*

(1) *$E \left(T(p,p) \right)$ and $E \left(T(p) \right)$ give naturally $\mathbb{Q}$-linear endomorphisms of V.*

(2) *For a prime $l \neq p$, the congruence relation*

$$E \left(T(p) \right) = F_p + p \left(F_p^{-1} \circ E \left(T(p,p) \right) \right)$$

holds on $V \otimes \mathbb{Q}_l$.

Proof. The assertion (1) follows from Propositions 3.3 and 3.4, and then we prove (2). By [KM, Theorem 10.12.2], $\overline{\rho}_1 : \overline{M}_{\zeta,p} \to \overline{M}_\zeta$ is flat, and hence by the geometric congruence relation,

$$\overline{\rho}_2 \circ (\overline{\rho}_1)^{-1} = \varphi_p + p \cdot \left(\varphi_p^{-1} \circ T(p,p) \right)$$

holds on $\overline{M}_\zeta \otimes \overline{\mathbb{F}}_p$, where φ_p is the p-th power map. Since C_ζ is finite, étale over R_n and consists of R_n-rational points,

$$\overline{\rho}_2 \circ (\overline{\rho}_1)^{-1} = \varphi + p \cdot \left(\varphi^{-1} \circ T(p,p) \right)$$

holds on C_ζ, where $\varphi \in \mathrm{Gal} \left(\mathbb{Q}(\zeta)/\mathbb{Q} \right)$ is a Frobenius automorphism at p. Therefore, the proof of Proposition 3.4 implies (2). $\square$

4. Decomposition by Hecke action

4.1. In this section, we introduce a theory of Hecke operators which gives a motivic decomposition of the elliptic modular motive with congruence relation. Let $\mathcal{H}$ be the Hecke algebra which is defined as a $\mathbb{Z}$-algebra generated by double coset classes by $\Gamma(n)$ of $T(p)$, $T(p,p)$ for primes p not dividing n. Then it is shown in [Sm] that $\mathcal{H}$ becomes a commutative ring.

PROPOSITION 4.1. *There exists a unique $\mathbb{Z}$-algebra homomorphism $\mathcal{H} \to \mathrm{End}(V)$ sending $T(p)$ and $T(p,p)$ to $E\left(T(p)\right)$ and $E\left(T(p,p)\right)$ respectively for primes p not dividing n.*

Proof. The assertion follows from Propositions 3.1, 3.3, 3.4 and that for two Hecke correspondences, their product in $\mathcal{H}$ corresponds to that as an algebraic correspondence. $\square$

We call the images of the above $\mathcal{H} \to \mathrm{End}(V)$ *Hecke operators* in our theory. Let $T(g) = \tau_1 \times \tau_2$ be the Hecke correspondence $T(p)$ or $T(p,p)$ for a prime p not dividing n, and put $d = \det(g)$. Then for any cusp $\beta \in \overline{Z}_\zeta - Z_\zeta$, the $\mathbb{Q}$-algebra homomorphism $A_N(\iota_\beta) : A_N\left(\overline{M}_\zeta(\mathbb{C}); \tau_1(\beta)\right) \to A_N\left(\overline{M}_{\zeta^d}(\mathbb{C}); \tau_2(\beta)\right)$ is defined in 3.2, and our $T(g)$ becomes these sum, where β runs through the scheme theoretic fiber of a cusp in C_ζ.

THEOREM 4.2. *Any Hecke operator is commutative with the Galois action on $V \otimes \mathbb{Q}_l$, and preserves the weight and Hodge filtrations of $V \otimes \mathbb{C}$.*

Proof. The former assertion follows from Proposition 3.3 (3), and we will show the second one. Let $T(g) = \tau_1 \times \tau_2$ be the Hecke correspondence $T(p)$ or $T(p,p)$ for a prime p not dividing n, and put $d = \det(g)$. Then for any cusp $\beta \in \overline{Z}_\zeta - Z_\zeta$ and $i = 1, 2$, by Proposition 3.2,

$$A_N(\tau_i) : A_N\left(\overline{Z}_\zeta(\mathbb{C}); z\right) \to A_N\left(\overline{M}_{\zeta_i}(\mathbb{C}); \tau_i(z)\right)$$

is surjective and preserves the weight filtrations. Further, $\tau_i : \overline{Z}_\zeta(\mathbb{C}) \to \overline{M}_{\zeta_i}(\mathbb{C})$ is a holomorphic map, and hence $A_N(\tau_i)$ preserves the Hodge decompositions. Therefore, $A_N(\tau_i)$ gives a surjection between each pair of the Hodge components. Since $A_N(\iota_\beta) \circ A_N(\tau_1) = A_N(\tau_2)$ by Proposition 3.3, $A_N(\iota_\beta)$ preserves the Hodge decompositions of $V \otimes \mathbb{C}$, and hence their sum $T(g)$ has the same property. $\square$

The following corollary gives a solution to a problem posed by Manin [M2, 3.3] for cusp forms of weight 2.

COROLLARY 4.3. *Let B_N be as in 2.2 which is the dual space of $V \otimes \mathbb{C}$ with action of $\mathcal{H}$. Then the subspace of B_N spanned by iterated Shimura integrals*

$$\int \omega_1 \cdots \omega_N \ \left(\omega_i : holomorphic \ 1\text{-}forms \ on \ \overline{M}_\zeta(\mathbb{C}) \right)$$

is stable under the action of $\mathcal{H}$.

Proof. Iterated integrals of holomorphic 1-forms are homotopy functional, and by Theorem 4.2, the action of $\mathcal{H}$ preserves the Hodge filtration of B_N, especially its subspace $F^N(B_N)$. $\square$

4.2. Since V is finite dimensional over $\mathbb{Q}$, the eigenvalues of each Hecke operator are in $\overline{\mathbb{Q}}$. For each representation $\varepsilon : \mathcal{H} \to \overline{\mathbb{Q}}$, i.e., $\mathbb{Z}$-algebra homomorphism, we define the *Hecke component* for ε as the subspace of $V_{\overline{\mathbb{Q}}} = V \otimes \overline{\mathbb{Q}}$ given by

$$V_{\overline{\mathbb{Q}}}(\varepsilon) = \bigcap_{h \in \mathcal{H}} \left(\bigcup_{m \in \mathbb{N}} \mathrm{Ker}\left((h - \varepsilon(h))^m \right) \right).$$

Then by the commutativity of $\mathcal{H}$, $V_{\overline{\mathbb{Q}}}$ is decomposed to the direct sum of the Hecke components:

$$V_{\overline{\mathbb{Q}}} = \bigoplus_\varepsilon V_{\overline{\mathbb{Q}}}(\varepsilon),$$

and hence by Theorems 4.2 and 3.5 (2), we have:

THEOREM 4.4.

(1) $V_{\overline{\mathbb{Q}}}(\varepsilon) \otimes \overline{\mathbb{Q}}_l$ *has Galois action by* $\sigma \otimes \mathrm{id}_{\overline{\mathbb{Q}}_l}$ $\left(\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q} \right) \right)$ *on* $(V \otimes \mathbb{Q}_l) \otimes \overline{\mathbb{Q}}_l$, *and* $V_{\overline{\mathbb{Q}}}(\varepsilon) \otimes \mathbb{C}$ *has mixed Hodge structure which are compatible with the motivic structure on V.*

(2) *For a prime $l \neq p$, the congruence relation*

$$E\left(T(p) \right) = F_p + p \left(F_p^{-1} \circ E\left(T(p, p) \right) \right)$$

holds on $V_{\overline{\mathbb{Q}}}(\varepsilon) \otimes \overline{\mathbb{Q}}_l$.

4.3. We study the mixedness of V in the category of motives over $\overline{\mathbb{Q}}$, where morphisms are considered as $\overline{\mathbb{Q}}$-linear homomorphisms compatible with Galois action and with weight and Hodge filtrations. For each $\zeta \in P_n$ and $\alpha \in C_\zeta$, let $I(\zeta, \alpha)$ be the augmentation ideal of $\mathbb{Q}\left[\pi_1\left(\overline{M}_\zeta(\mathbb{C}); \alpha\right)\right]$, and put

$$I^m = \bigoplus_{\zeta, \alpha} I(\zeta, \alpha)^m$$

which make a decreasing sequence of $\mathbb{Q}$-subspaces of V. By construction, the action of $\mathcal{H}$ preserves this filtration.

THEOREM 4.5. *For all integers $1 < l < m \leq N$, the exact sequence*

$$0 \to \left(I^l/I^m\right) \otimes \overline{\mathbb{Q}} \to (I/I^m) \otimes \overline{\mathbb{Q}} \to \left(I/I^l\right) \otimes \overline{\mathbb{Q}} \to 0$$

splits as motives over $\overline{\mathbb{Q}}$.

Proof. Let p be a prime not dividing n. Then by a famous result of Weil,

$$I(\zeta, \alpha)/I(\zeta, \alpha)^2 \cong H_1\left(\overline{M}_\zeta(\mathbb{C}); \mathbb{Q}\right)$$

has pure weight -1, i.e., after scalar-extended to $\mathbb{Q}_l$, a p^dth power Frobenius homomorphism has eigenvalues with absolute value $p^{d/2}$. Hence by the surjectivity of the natural $\mathbb{Q}$-linear homomorphism

$$\left(I(\zeta, \alpha)/I(\zeta, \alpha)^2\right)^{\otimes m} \to I(\zeta, \alpha)^m/I(\zeta, \alpha)^{m+1},$$

each I^m/I^{m+1} has pure weight $-m$. As seen above, $I \otimes \overline{\mathbb{Q}}$ is the direct sum of

$$I_{\overline{\mathbb{Q}}}(\varepsilon) = V_{\overline{\mathbb{Q}}}(\varepsilon) \cap \left(I \otimes \overline{\mathbb{Q}}\right)$$

as a motive over $\overline{\mathbb{Q}}$. Therefore, to show the assertion, we may prove that $I_{\overline{\mathbb{Q}}}(\varepsilon) \cap \left(I^m \otimes \overline{\mathbb{Q}}\right)$ becomes either $\{0\}$ or $I_{\overline{\mathbb{Q}}}(\varepsilon)$ for any ε and $m \geq 2$ since in this case, $\left(I^l/I^m\right) \otimes \overline{\mathbb{Q}}$ is isomorphic to the direct sum of $I_{\overline{\mathbb{Q}}}(\varepsilon)$ which are contained in $I^l \otimes \overline{\mathbb{Q}}$ and not contained in $I^m \otimes \overline{\mathbb{Q}}$. Assume on the contrary that

$$\{0\} \subsetneq W_1 = I_{\overline{\mathbb{Q}}}(\varepsilon) \cap (I^m \otimes \overline{\mathbb{Q}}) \subsetneq W_2 = I_{\overline{\mathbb{Q}}}(\varepsilon)$$

for some ε and $m \geq 2$. Let e_1 and e_2 be the eigenvalues of the Frobenius F_p on $W_1 \otimes \overline{\mathbb{Q}}_l$ and on $(W_2/W_1) \otimes \overline{\mathbb{Q}}_l$ respectively. Then by Theorem 4.4 (2), e_1 and e_2 are

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the roots of $x^2 - \varepsilon(T(p))x + p \cdot \varepsilon(T(p,p)) = 0$ such that $p^{1/2} \le |e_2| < p^{m/2} \le |e_1|$. Therefore, $|p \cdot \varepsilon(T(p,p))| = |e_1 e_2| > p$ which contradicts with that $E(T(p,p))$ is an automorphism of V with finite order. This completes the proof. $\square$

THEOREM 4.6. *Fix $\zeta \in P_n$ and $\alpha \in C_\zeta$. Then for any $\gamma \in \pi_1 \left(\overline{M}_\zeta(\mathbb{C}); \alpha \right)$ and all holomorphic 1-forms $\omega_1, ..., \omega_m$ on $\overline{M}_\zeta(\mathbb{C})$, there exist a positive integer l, $d_j \in \overline{\mathbb{Q}}$ and $\delta_{ij} \in H_1 \left(\overline{M}_\zeta(\mathbb{C}); \mathbb{Z} \right)$ $(1 \le i \le m, 1 \le j \le l)$ such that*

$$\int_\gamma \omega_1 \cdots \omega_m = \sum_{j=1}^{l} d_j \left(\prod_{i=1}^{m} \int_{\delta_{ij}} \omega_i \right).$$

Proof. We use results in [H2, §6] with some extension. Take $\{g(\zeta, \alpha)_i\}_i \subset I(\zeta, \alpha)$ giving a basis of I/I^m which also provides a section of the projection $I/I^{m+1} \to I/I^m$. For each element $w_1 \otimes \cdots \otimes w_m$ $\left(w_i : \text{closed 1-forms on } \overline{M}_\zeta(\mathbb{C}) \right)$ of a basis of

$$\mathrm{Hom}\left(I(\zeta, \alpha)^m / I(\zeta, \alpha)^{m+1}, \mathbb{Q} \right) \otimes \mathbb{C} \hookrightarrow H_1 \left(\overline{M}_\zeta(\mathbb{C}); \mathbb{C} \right)^{\otimes m}$$

with Hodge filtration, take a set $\left\{ u_{j1}, ..., u_{jn_j} \right\}_j$ $(n_j < m)$ of 1-forms on $\overline{M}_\zeta(\mathbb{C})$ such that

$$\int w_1 \cdots w_m + \sum_j \int u_{j1} \cdots u_{jn_j}$$

is a homotopy functional iterated integral. Then the bilinear form

$$\langle \gamma - 1, w_1 \otimes \cdots \otimes w_m \rangle$$
$$= \begin{cases} \displaystyle\int w_1 \cdots w_m + \sum_j \int u_{j1} \cdots u_{jn_j} & \left(\gamma \in \pi_1 \left(\overline{M}_\zeta(\mathbb{C}); \alpha \right) \right), \\ 0 & \left(\gamma \in \pi_1 \left(\overline{M}_{\zeta'}(\mathbb{C}); \alpha' \right) \text{ for } (\zeta', \alpha') \ne (\zeta, \alpha) \right) \end{cases}$$

gives a retraction of the inclusion $(I^m/I^{m+1}) \otimes \mathbb{C} \to (I/I^{m+1}) \otimes \mathbb{C}$ which preserves their Hodge filtrations. Therefore, the extension data of

$$0 \to I^m/I^{m+1} \to I/I^{m+1} \to I/I^m \to 0$$

as their mixed Hodge structures over $\overline{\mathbb{Q}}$ is given as an element of

$$\frac{\mathrm{Hom}\left(I/I^m, I^m/I^{m+1} \right) \otimes \mathbb{C}}{F^0 \left(\mathrm{Hom}\left(I/I^m, I^m/I^{m+1} \right) \otimes \mathbb{C} \right) + \mathrm{Hom}\left(I/I^m, I^m/I^{m+1} \right) \otimes \overline{\mathbb{Q}}}$$

which sends $g(\zeta, \alpha)_i$ to

$$w_1 \otimes \cdots \otimes w_m \mapsto \langle g(\zeta, \alpha)_i, w_1 \otimes \cdots \otimes w_m \rangle.$$

Since $\omega_1, ..., \omega_m$ are holomorphic, $\int \omega_1 \cdots \omega_m$ is homotopy functional, and any element of $F^0 \left(\mathrm{Hom} \left(I/I^m, I^m/I^{m+1} \right) \otimes \mathbb{C} \right)$ sends each element of I/I^m to

$$F^m \left(\mathrm{Hom} \left(I^m/I^{m+1}, \mathbb{Q} \right) \otimes \mathbb{C} \right) \ni \omega_1 \otimes \cdots \otimes \omega_m \mapsto 0$$

because $F^m \left(\mathrm{Hom} \left(I/I^m, \mathbb{Q} \right) \otimes \mathbb{C} \right) = \{0\}$. Therefore, by Theorem 4.5,

$$\omega_1 \otimes \cdots \otimes \omega_m \mapsto \langle g(\zeta, \alpha)_i, \omega_1 \otimes \cdots \otimes \omega_m \rangle$$

belongs to $(I^m/I^{m+1}) \otimes \overline{\mathbb{Q}}$, and hence becomes a $\overline{\mathbb{Q}}$-linear sum of $\prod_{i=1}^{m} \int_{\delta_i} \omega_i$ for some $\delta_i \in H_1 \left(\overline{M}_\zeta(\mathbb{C}); \mathbb{Z} \right)$. $\square$

Theorem 4.6 and calculation of iterated Shimura integrals in [M1] imply the following:

COROLLARY 4.7.   *For cusp forms*

$$\omega_i = \sum_{l=1}^{\infty} c_i(l) e^{2\pi\sqrt{-1}l\tau/n} \ (1 \le i \le m)$$

of weight 2 and level n, and for $a \in \Gamma(n) \left(\sqrt{-1} \cdot \infty \right)$, the multiple L-value

$$\sum_{0 < l_1 < \cdots < l_m} \frac{c_1(l_m - l_{m-1}) \cdots c_m(l_1)}{l_m \cdots l_1} e^{2\pi\sqrt{-1}l_m a/n}$$

becomes a $\overline{\mathbb{Q}}$-linear sum of the products

$$\prod_{i=1}^{m} \left(\sum_{l_i=1}^{\infty} \frac{c_i(l_i)}{l_i} e^{2\pi\sqrt{-1}l_i a_i/n} \right)$$

of L-values for some $a_i \in \Gamma(n) \left(\sqrt{-1} \cdot \infty \right)$.

Remark. As is seen in the above consideration, "$\overline{\mathbb{Q}}$" in Theorems 4.5, 4.6 and Corollary 4.7 can be replaced with a finite extension of $\mathbb{Q}$ over which the decomposition $I = \bigoplus_\varepsilon I(\varepsilon)$.

5. Noncommutative modular symbols

5.1. In this section, we express the Betti realization of the elliptic modular motive as the space of noncommutative modular symbols, and describe the action of Hecke operators on this space. This description will be useful in computing our Hecke action. For an element $\zeta \in P_n$ and a cusp $\alpha \in C_\zeta = \mathbb{P}^1(\mathbb{Q})/\Gamma(n)$ on $\overline{M}_\zeta(\mathbb{C}) = \overline{H}/\Gamma(n)$, let $\mathcal{M}_N(\zeta, \alpha)$ be the space of Nth truncated (cuspidal) *noncommutative modular symbols* which is defined to be a $\mathbb{Q}$-algebra generated by symbols

$$[a, b] \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}),$$

where a, b belongs to α (: considered as a $\Gamma(n)$-orbit), satisfying the relations:

- $[a, a] = 1$,

- $[a, b] \cdot [b, c] = [a, c]$,

- $[a, b] = [\gamma(a), \gamma(b)]$ for any $\gamma \in \Gamma(n)$,

- $[a, \gamma(a)] = 1$ for any parabolic $\gamma \in \Gamma(n)$,

- $([a_1, b_1] - 1) \cdot ([a_2, b_2] - 1) \cdot \cdots \cdot ([a_{N+1}, b_{N+1}] - 1) = 0$.

By these relations, $\mathcal{M}_N(\zeta, \alpha)$ is generated by $[a, b]$ for a fixed $a \in \mathbb{P}^1(\mathbb{Q})$ as a $\mathbb{Q}$-vector space, and $\mathcal{M}_1(\zeta, \alpha)/(\mathbb{Q} \cdot 1)$ becomes the space of usual modular symbols $\{a, b\}$ by the correspondence $[a, b] = 1 + \{a, b\}$. Furthermore, by associating $[a, b]$ with the image in $\overline{M}(\mathbb{C})$ of an oriented path from a to b in H, we have

$$\mathcal{M}_N(\zeta, \alpha) \cong A_N\left(\overline{M}_\zeta(\mathbb{C}); \alpha\right).$$

5.2. Under the above isomorphism, we describe the action of Hecke operators on noncommutative modular symbols. For a Hecke correspondence $T(g) = T(p), T(p, p)$ for a prime p not dividing n, put $d = \det(g)$, $\Gamma(n, g) = \Gamma(n) \cap g^{-1}\Gamma(n)g$. Further, fix $a_0 \in \mathbb{P}^1(\mathbb{Q})$, and put $a_j = \gamma_j(a_0)$ for a coset decomposition $\Gamma(n) = \bigsqcup_j \Gamma(n, g)\gamma_j$. Then for each $[a_j, b] \in \mathcal{M}_N(\zeta, \alpha)$, take $\gamma \in \Gamma(n)$ and a positive integer t such that $b = \gamma(a_j)$ and $\gamma^t \in \Gamma(n, g)$, and define

$$
\begin{aligned}
E(g, a_j)\left([a_j, b]\right) &= E(g, a_j)\left(\exp\left(\frac{1}{t}\log\left[a_j, \gamma^t(a_j)\right]\right)\right) \\
&= \exp\left(\frac{1}{t}\log\left[g(a_j), (g\gamma^t)(a_j)\right]\right).
\end{aligned}
$$

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Since $g\gamma^t \in \Gamma(n)g$, $E(g, a_j)\left([a_j, b]\right)$ belongs to $\mathcal{M}_N\left(\zeta^d, \alpha_j\right)$, where α_j is the cusp given by $g(a_j)$. Then by Proposition 3.3 and its proof, we have:

THEOREM 5.1. *Let* β_j *be the cusp on* $\overline{H}/\Gamma(n, g)$ *given by* a_j. *Then* $E(g, a_j) = \phi \circ \iota_{\beta_j}$ *on* $\mathcal{M}_N(\zeta, \alpha) \cong A_N\left(\overline{M}_\zeta(\mathbb{C}); \alpha\right)$, *and hence* $E(g, a_j)$ *gives a well-defined* $\mathbb{Q}$-*algebra homomorphism.*

COROLLARY 5.2. $E(T(g)) = \sum_j E(g, a_j)$ *on* $V = \bigoplus_{\zeta, \alpha} \mathcal{M}_N(\zeta, \alpha)$, *where* ζ, α *run through* P_n, C_ζ *respectively.*

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